ON GENERALIZED pseudo-projective curvature tensor on para-kenmotsu manifolds

Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 1(63), No. 1 - 2021, 143-160
https://doi.org/10.31926/but.mif.2021.1.63.1.11

ON GENERALIZED PSEUDO-PROJECTIVE CURVATURE TENSOR OF PARA-KENMOTSU MANIFOLDS

A. GOYAL [1], G. PANDEY [2], M. K. PANDEY [3] and T. RAGHUWANSHI*[4]

Abstract

The object of the present paper is to generalize pseudo-projective curvature tensor of para-Kenmotsu manifold with the help of a new generalized (0,2) symmetric tensor Z introduced by Mantica and Suh. Various geometric properties of generalized pseudo-projective curvature tensor of para-Kenmotsu manifold have been studied. It is shown that a generalized pseudo-projectively ϕ-symmetric para-Kenmotsu manifold is an Einstein manifold.

2010 *Mathematics Subject Classification:* 53C15, 53C25.

Key words: Pseudo-projective curvature tensor, para-Kenmotsu manifold, Einstein manifold, η-Einstein manifold, Generalized pseudo-projective curvature tensor.

1 Introduction

The projective tensor is one of the major curvature tensors. The study of pseudo-projective curvature tensor has been a very attractive field for investigations in the past decades. A tensor field $\overline{P}$ was defined and studied in 2002 by Bhagwat Prasad [18] on a Riemannian manifold of dimension n, which includes projective curvature tensor P. This tensor field $\overline{P}$ referred to as pseudo-projective curvature tensor. In 2011, H.G. Nagaraja and G. Somashekhara [14] extended pseudo-projective curvature tensor in Sasakian manifolds. After 2012, the pseudo-projective curvature tensor analysis in LP-Sasakian manifolds was resumed by

[1]Department of Mathematics, University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Bhopal, Madhya Pradesh 462033, India, e-mail: anil_goyal03@rediffmail.com
[2]Department of Mathematics, Govt. Tulsi College, Anuppur, Madhya Pradesh 484224, India, e-mail: math.giteshwari@gmail.com
[3]Department of Mathematics, University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Bhopal, Madhya Pradesh 462033, India, e-mail: mkp_apsu@rediffmail.com
[4]*Corresponding author, Department of Mathematics, University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Bhopal, Madhya Pradesh 462033, India, e-mail: teerathramsgs@gmail.com

E-mail: hegtdcano@mp.gov.in 9893076404

Y.B. Maralabhavi and G.S. Shivaprasanna [12]. In 2016, S. Mallick, Y.J. Suh and U.C. De [11] defined and studied a space time with pseudo-projective curvature tensor. Subsequently, several researchers performed a study of pseudo-projective curvature tensor in a number of directions, such as [4, 5, 13, 15, 17, 21, 22]. The pseudo-projective curvature tensor is defined by [18]

$$\begin{aligned}\overline{P}(X,Y,U) =& aR(X,Y,U) + b[S(Y,U)X - S(X,U)Y] \\ &- \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,U)X - g(X,U)Y],\end{aligned} \tag{1}$$

where a and b are constants such that a, b $\neq 0$ and R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature tensor.

The notion of an almost para-contact manifold was introduced by I. Sato [19]. Since the publication of [26], paracontact metric manifolds have been studied by many authors in recent years. The importance of para-Kenmotsu geometry, have been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-Kähler manifolds and its role in semi-Riemannian geometry and mathematical physics [3, 7, 8, 20].

In this paper, we consider the generalized pseudo-projective curvature tensor of para-Kenmotsu manifolds and study some properties of generalized pseudo-projective curvature tensor. The organisation of the paper is as follows: After preliminaries on para-Kenmotsu manifold in Section 2, we describe briefly the generalized pseudo-projective curvature tensor on para-Kenmotsu manifold in Section 3 and also we study some properties of generalized pseudo-projective curvature tensor in para-Kenmotsu manifold. In Section 4, we study a generalized pseudo-projectively semi-symmetric para-Kenmotsu manifold is an η Einstein manifold. Further in the Section 5, we show that a generalized pseudo-projectively Ricci semi-symmetric para-Kenmotsu manifold is either Einstein manifold or $\psi = \frac{an(n-1)+ra+br(n-1)}{bn(n-1)}$ on it. In the last section we show that the generalized pseudo-projectively ϕ-symmetric para-Kenmotsu manifold is an Einstein manifold.

2 Preliminaries

An n-dimensional differentiable manifold M^n is said to have almost para-contact structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \tag{2}$$

$$\eta(\phi X) = 0, \tag{3}$$

$$\phi(\xi) = 0, \tag{4}$$

and

$$\eta(\xi) = 1. \tag{5}$$

PRINCIPAL
Govt. Tulsi College Anuppur
Distt. Anuppur (M.P.)

E-mail: hegtdcano@mp.gov.in 9893076404

A differentiable manifold with almost para-contact structure (ϕ, ξ, η) is called an almost para-contact manifold. Further, if the manifold M^n has a semi-Riemannian metric g satisfying

$$\eta(X) = g(X, \xi) \tag{6}$$

and

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{7}$$

Then the structure (ϕ, ξ, η, g) satisfying conditions (2) to (7) is called an almost para-contact Riemannian structure and the manifold M^n with such a structure is called an almost para-contact Riemannian manifold [1, 19].

Now we briefly present an account of an analogue of the Kenmotsu manifold in paracontact geometry which will be called para-Kenmotsu.

Definition 1. *The almost paracontact metric structure (ϕ, ξ, η, g) is para-Kenmotsu should this relation hold[2, 16], if the Levi-Civita connection ∇ of g satisfies $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any $X, Y \in \mathfrak{X}(M)$.*

On a para-Kenmotsu manifold [2, 20], the following relations hold:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{8}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{9}$$

$$\eta(R(X, Y, Z)) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{10}$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$R(X, \xi, Y) = -R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \tag{12}$$

$$S(\phi X, \phi Y) = -(n-1)g(\phi X, \phi Y), \tag{13}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{14}$$

$$Q\xi = -(n-1)\xi, \tag{15}$$

$$r = -n(n-1), \tag{16}$$

for any vector fields X, Y, Z , where Q is the Ricci operator that is $g(QX, Y) = S(X, Y)$, S is the Ricci tensor and r is the scalar curvature.

In A. M. Blaga [2]. gave an example on para-Kenmotsu manifold:

Example 1. *We consider the three dimensional manifold $M^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields*

$$e_1 := \frac{\partial}{\partial x}, e_2 := \frac{\partial}{\partial y}, e_3 := -\frac{\partial}{\partial z}$$

are linearly independent at each point of the manifold.
Define

$$\phi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \xi := -\frac{\partial}{\partial z}, \eta := -dz.$$

PRINCIPAL
Govt. Tulsi College Anuppur
Distt. Anuppur (M.P.)

OFFICE, PRINCIPAL GOVERNMENT TULSI COLLEGE, ANUPPUR
Affiliated to Awadhesh Pratap Singh University Rewa (MP)
Registered Under Section 2 (F) & 12 (B) of UGC Act

E-mail: hegtdcano@mp.gov.in 9893076404

$$g := dx \otimes dx - dy \otimes dy + dz \otimes dz.$$

Then it follows that

$$\phi e_1 = e_2, \phi e_2 = e_1, \phi e_3 = 0,$$
$$\eta(e_1) = 0, \eta(e_2) = 0, \eta(e_3) = 1.$$

Let ∇ *be the Levi-Civita connetion with respect to metric* g*. Then, we have*

$$[e_1, e_2] = 0, [e_2, e_3] = 0, [e_3, e_1] = 0$$

The Riemannian connection ∇ *of the metric* g *is deduced from Koszul's formula*

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]).$$

Then Koszul's formula yields

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e_1,$$
$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = e_3, \nabla_{e_2} e_3 = e_2,$$
$$\nabla_{e_3} e_1 = e_1, \nabla_{e_3} e_2 = e_2, \nabla_{e_3} e_3 = 0.$$

These results shows that the manifold satisfies

$$\nabla_X \xi = X - \eta(X)\xi,$$

for $\xi = e_3$*. Hence the manifold under consideration is para-Kenmotsu manifold of dimension three.*

A para-Kenmotsu manifold is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{17}$$

for the vector fields X, Y, where a and b are functions on M^n.

3 Generalized pseudo-projective curvature tensor of para-Kenmotsu manifold

In this section, we give a brief account of generalized pseudo-projective curvature tensor of para-Kenmotsu manifold and study various geometric properties of it.

The pseudo-projective curvature tensor of para-Kenmotsu manifold M^n is given by the following relation:

$$\overline{P}(X, Y, U) = aR(X, Y, U) + b[S(Y, U)X - S(X, U)Y] - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y, U)X - g(X, U)Y], \tag{18}$$

PRINCIPAL
Govt. Tulsi College Anuppur
Distt. Anuppur (M.P.)

Also, the type (0, 4) tensor field $'\overline{P}$ is given by

$$'\overline{P}(X,Y,U,V) = a'R(X,Y,U,V) + b[S(Y,U)g(X,V) - S(X,U) g(Y,V)] - \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,U)g(X,V) - g(X,U)g(Y,V)], \tag{19}$$

where

$$'\overline{P}(X,Y,U,V) = g(\overline{P}(X,Y,U),V)$$

and

$$'R(X,Y,U,V) = g(R(X,Y,U),V)$$

for the arbitrary vector fields X, Y, U, V.

Differentiating covariantly with respect to W in equation (18), we get

$$(\nabla_W\overline{P})(X,Y)U) = a(\nabla_W R)(X,Y)U) + b[(\nabla_W S)(Y,U)X -(\nabla_W S)(X,U)Y] - \frac{dr(W)}{n}\left(\frac{a}{n-1}+b\right)[g(Y,U)X - g(X,U)Y]. \tag{20}$$

Divergence of pseudo-projective curvature tensor in equation (18) is given by

$$(div\overline{P})(X,Y)U) = a(divR)(X,Y)U) + b[(\nabla_X S)(Y,U) -(\nabla_Y S)(X,U)] - (divr)\left[\frac{a+b(n-1)}{n(n-1)}\right][g(Y,U)div(X) -g(X,U)div(Y)]. \tag{21}$$

But

$$(divR)(X,Y)U) = (\nabla_X S)(Y,U) - (\nabla_Y S)(X,U). \tag{22}$$

From equations (21) and (22), we have

$$(div\overline{P})(X,Y)U) = (a+b)[(\nabla_X S)(Y,U) - (\nabla_Y S)(X,U)] - (divr) \left[\frac{a+b(n-1)}{n(n-1)}\right][g(Y,U)div(X) - g(X,U)div(Y)]. \tag{23}$$

Definition 2. *An almost paracontact structure (ϕ,ξ,η,g) is said to be locally pseudo-projectively symmetric if*

$$(\nabla_W\overline{P})(X,Y,U) = 0, \tag{24}$$

for all vector fields $X, Y, U, W \in T_pM^n$.

Definition 3. *An almost paracontact structure (ϕ,ξ,η,g) is said to be locally pseudo-projectively ϕ-symmetric if*

$$\phi^2((\nabla_W\overline{P})(X,Y,U)) = 0, \tag{25}$$

for all vector fields X, Y, U, W orthogonal to ξ.